Shape Analysis for Redistricting
modern geometry meets modern politics
Zachary Schutzman

University of Pennsylvania & Metric Geometry and Gerrymandering Group

Hyperbolic Lunch, U. of Toronto Mathematics

February 21st, 2019
This is joint work with...

**Total Variation Isoperimetric Profiles**
https://arxiv.org/abs/1809.07943

Daryl DeFord
Hugo Lavenant
Justin Solomon

**Graph Laplacians**
Emilia Alvarez
Daryl DeFord
James Murphy
Justin Solomon

**Discrete Compactness**
Assaf Bar-Natan
Moon Duchin
Adriana Rogers

**Curve-Shortening Flow**
https://zachschutzman.com/distflow

Emilia Alvarez
Daryl DeFord
Michelle Feng
Patrick Girardet
Natalia Hajlasz
Eduardo Chavez Heredia
Lorenzo Najt
Sloan Nietert
Aidan Perreault
Justin Solomon
WHAT IS COMPACTNESS?
Compactness is ...

Vaguely, it’s supposed to describe the niceness of the shape of a district.
Compactness is in the discourse.

America’s most gerrymandered congressional districts

Michigan has districts in shapes that even a flexible salamander would be incapable of contorting itself into. To be sure, there are perfectly legitimate reasons for Democratic-leaning Ohio’s wonky districts.

North Carolina

Winston-Salem, Greensboro, Durham, Raleigh

Ohio’s US District 9
Compactness is in the law

Legislative Apportionment Commission shall attempt to form functionally contiguous and compact territories. For purposes of this section, a

Iowa law provides that congressional and legislative districts should be reasonably compact in form. As noted previously, the requirement to establish

(i) Each congressional district shall consist of areas of convenient territory contiguous by land. Areas that meet only at points of adjoining corners are not contiguous.
Compactness is poorly defined

contiguous and compact territories. For purposes of this section, a "functionally contiguous and compact territory" is one that facilitates representation by minimizing impediments to travel within the district.

In order to compare the relative compactness of two or more districts or of two or more alternative redistricting plans, the Code provides that two measures of compactness, length-width compactness and perimeter compactness, shall be used.

(vii) Compactness shall be determined by circumscribing each district within a circle of minimum radius and measuring the area, not part of the Great Lakes and not part of another state, inside the circle but not inside the district.
The measures are basic Polsby-Popper

\[ 0 < PP(\Omega) = \frac{4\pi \cdot \text{Area}(\Omega)}{\text{Perim}^2(\Omega)} \leq 1 \]
The measures are basic Polsby-Popper

scale-free

isoperimetric

circle

loves circles

sensitive
The measures are basic

Bounding regions

\[ f(\Omega) = \frac{\text{Area}(\Omega)}{\text{Area}(B(\Omega))} \]
The measures are basic Bounding regions

\( B \) can be

- Circle [Reock]
- Square [Square Reock]
- Convex hull [Convex hull]
- Ellipse, rectangle
- Axis-aligned ellipse, rectangle
  ...

The measures are basic

Bounding regions

scale-free
not sensitive at boundary

inconsistent
good
interpretation?
The measures are basic Miscellany

Largest inscribed circle

Just the perimeter

Longest axis by greatest orthogonal width

Population-weighted versions

Reciprocal of Polsby-Popper
What's the Takeaway?

The geometry is important, and a lot of geometry has been done in the last 2000 years. So, let's use it. But, maybe we should care a little less.
This talk:

The case for multiscale methods
‘Continuous’ definitions
  Isoperimetric profiles/total variation
  Curve-shortening flow
‘Discrete’ definitions
  Constructing a dual graph
  Discrete analogues
  Graph spectrum
  Discrete curvature?

You should ask me questions
What's the dream?

**Computable**: we should have a good algorithm to find the measure

**Stable**: similar shapes should have similar scores

**Informative**: the score should say something about the geometry

**Explainable**: it should be easy to tell someone what’s going on
CONTINUOUS
METHODS
Isoperimetric profiles

"Total Variation Isoperimetric Profiles" (2018), DeFord, Lavenant, Schutzman, & Solomon

“For all times $t \in (0, 1]$, what is the smallest perimeter of any inscribed subregion of $\Omega$ which fills a $t$-fraction of the area?”

Gives you a function or a curve or a vector from your shape.
What's so cool about it?

\[ t = 1 \] recovers the Polsby-Popper score
Some basic algebra lets you get the largest inscribed circle
Stable under perturbations
The function and its derivative tell you some stuff about the shape at different resolutions
Formalization

\[ TV[f] = \int_{\mathbb{R}^n} \| \nabla f \|_2 \, dx. \]

\[ \text{area}(\partial \Sigma) = TV[1_\Sigma]. \]

\[ I_\Omega(t) = \left\{ \begin{array}{l}
\inf_{f \in L^1(\mathbb{R}^n)} \quad TV[f] \\
\text{subject to} \\
\int_{\mathbb{R}^n} f(x) \, dx = t \\
0 \leq f \leq 1_\Omega \\
f(x) \in \{ 0, 1 \} \quad \forall x \in \mathbb{R}^n.
\end{array} \right. \]
\[
I_\Omega(t) = \left\{ \begin{array}{l}
\inf_{f \in L^1(\mathbb{R}^n)} \text{TV}[f] \\
\text{subject to} \\
\int_{\mathbb{R}^n} f(x) \, dx = t \\
0 \leq f \leq 1_\Omega \\
f(x) \in \{0, 1\} \quad \forall x \in \mathbb{R}^n.
\end{array} \right.
\]
Convexify!

\[ I_\Omega(t) = \begin{cases} \inf_{f \in L^1(\mathbb{R}^n)} TV[f] \\ \text{subject to} \\ \int_{\mathbb{R}^n} f(x) \, dx = t \\ 0 \leq f \leq 1_\Omega \\ f(x) \in \mathbb{I}([0, 1]) \quad \forall x \in \mathbb{R}^n. \end{cases} \]
$I_{\Omega}(t) = \left\{ \begin{array}{l} \inf_{f \in L^1(\mathbb{R}^n)} TV[f] \\ \text{subject to} \\ \int_{\mathbb{R}^n} f(x) dx = t \\ 0 \leq f \leq 1_{\Omega} \\ f(x) \in [0, 1] \quad \forall x \in \mathbb{R}^n. \end{array} \right.$
Convexify!

\[ I_\Omega(t) = \left\{ \begin{array}{ll}
\inf_{f \in L^1(\mathbb{R}^n)} & \text{TV}[f] \\
\text{subject to} & \int_{\mathbb{R}^n} f(x)dx = t \\
& 0 \leq f \leq 1_{\Omega}
\end{array} \right\} \]

Using some duality arguments, we show that this is the \textit{lower convex envelope} of the Isoperimetric profile. (See the paper)
See it in action!
See it in action!
See it in action!

<table>
<thead>
<tr>
<th>District 1</th>
<th>District 2</th>
<th>District 3</th>
<th>District 4</th>
<th>District 5</th>
<th>District 6</th>
<th>District 7</th>
<th>District 8</th>
<th>District 9</th>
<th>District 10</th>
<th>District 11</th>
<th>District 12</th>
<th>District 13</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
<td><img src="image13.png" alt="Image" /></td>
</tr>
</tbody>
</table>
See it in action!
Nice properties

satisfies three of our desiderata

good algorithms to compute

we can make it measure-aware

isoperimettricky
An Open Problem

Open Problem

The TV relaxation works in $\mathbb{R}^n$ (examples of $\mathbb{R}^3$ in the paper) and should work over any metric space where all the calculus stuff makes sense.

Is there a good algorithm to compute the isoperimetric profile in $\mathbb{R}^2$?
Curve-shortening flow

take a (closed) smooth curve in the plane
at each time step, at each point:

(1) find the curvature $\kappa$

(2) move a distance proportional to $\kappa$ ...

... in the direction normal to the tangent

(3) rescale the area
Curve-shortening flow

the perimeter shrinks
becomes a circle
in finite time

Record the PP score at each time
This assigns a function to a shape
What's so cool about it?

\[ t = 0 \] recovers the Polsby-Popper score monotonically decreasing in \( t \)
discretizes nicely
satisfies all four desiderata

The function and its derivative tell you some stuff about the shape at different resolutions
See it in action!

http://zachschutzman.com/distflow
Nice properties

satisfies our desiderata

easy to compute

discretizes nicely

isoperimetricky
DISCRETE METHODS
Constructing the dual graph
the districts are subgraphs
we can talk about ‘boundary’ and ‘interior’ nodes
there’s a natural metric to use
discrete polsby-popper
discrete convex hull
A quick illustration

Graphic adapted from Duchin & Tenner
What's the upside?

dual graphs have structure!
the sensitivity issue largely goes away
no longer depends on the $\mathbb{R}^2$ embedding

But, we know how to do more with graphs than just count vertices!
Graph Laplacian

Take a graph. Define the Laplacian $\mathcal{L}$ as the matrix with $-1$ in entry $ij$ if edge $ij$ is in the graph and $\deg(i)$ in entry $ii$. Zeros elsewhere.

This matrix is real and symmetric, so it’s positive semi-definite
Let’s consider its eigenvalues.
Laplacians

\[ \mathcal{L}_S = \begin{bmatrix} \vdots & \ddots & \vdots \end{bmatrix} \quad \mathcal{L}_P = \begin{bmatrix} [\mathcal{L}_{d_1}] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [\mathcal{L}_{d_n}] \end{bmatrix} \]

\( \mathcal{L}_P \) is \( \mathcal{L}_S \) with some edges deleted.

These two matrices ‘know’ almost all of the discrete geometry of a districting plan.
There’s a zero eigenvalue for each connected component
The second eigenvalue is no more than the vertex connectivity

Moral truth: the $k$th eigenvalue says something about how easy it is to cut the graph into $k$ pieces.
Laplace spectrum: large eigenvalues

The largest eigenvalue is less than the max degree
Summing in reverse, the degree sequence majorizes the eigenvalues
Kirchoff’s Matrix-Tree Theorem
Summing eigenvalues correlates very strongly with geometric compactness measures. Why?

Do these have any meaning as operators?

Is there meaning to the Laplace eigenvectors?
THANK YOU!