Graphs, Geometry, and Gerrymanders

a guide for the modern mathematician

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This is based on work with...

`gerrychain + trees`

`https://github.com/mggg/gerrychain`

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Metagraph

`https://mggg.org/metagraph`

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WHAT IS REDISTRICTING?

WHY DOES IT MATTER?

Redistricting

…is done regularly.

…occurs at a variety of levels.

…is baked into our system of government.

The Problem (Classical Version)

What is the ‘correct’ way to draw districts?

Is there even a good notion of ‘correct’?

Can we identify when a districting plan is ‘incorrect’?

Politicians draw political districts.
Incentives do not always align.

The Definition

Gerrymandering is the intentional drawing of electoral districts to favor or disfavor some political outcome.

A brief history of Canadian redistricting

Pre-1950s: big problem

1957: Manitoba tries something new

Now: Canada uses independent commissions for federal districts

Ridings within a province can differ in population by up to 25% legally, 15% in practice.

Recent Issues

Rural vs. Urban: 2006 PEI, 2012 Saskatchewan, 1991 Alberta

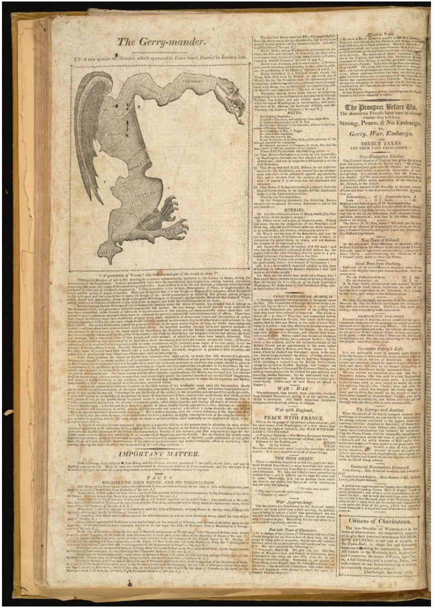
Right Now: Reducing the number of Toronto City Council wards

Where do people live?

In general, Canada's issue is *malapportionment*, not *gerrymandering*

Nationally, the smallest riding has about $\frac{1}{4}$ the population of the largest.

Gerry's Salamander



The Gerry-mander.

A new species of Monster, which appeared in Essex South District in January last.

IMPORTANT MATTER.

FACTS RELATIVE TO JOHN HENRY, AND HIS NEGOTIATION.

The Prospect Before Us.

The American People have now to choose whether they will have

Strong, Peace, & No Embargo,

or,

Gerry, War, Embargo,

and

DIRECT TAXES.

LET THEM TAKE THEIR CHOICE !!

WAR ! WAR !

PEACE WITH FRANCE.

THE IRON CHEST.

War Approaching.

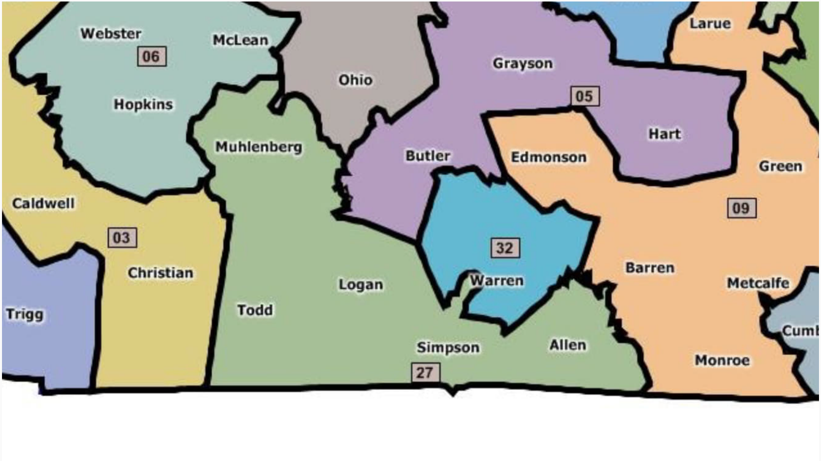
Citizens of Charlestown.

Gerry's Salamander

Tradition of Abuse

Draw incumbents in or out

Preserve an Incumbent



Tradition of Abuse

Draw incumbents in or out

Ohio 1880s

Ohio: Six Times in Twelve Years

12 Ninth Apportionment — 1878 to 1880
13 Tenth Apportionment — 1880 to 1882
14 Eleventh Apportionment — 1882 to 1884
15 Twelfth Apportionment — 1884 to 1886
16 Thirteenth Apportionment — 1886 to 1890
17 Fourteenth Apportionment — 1890 to 1892

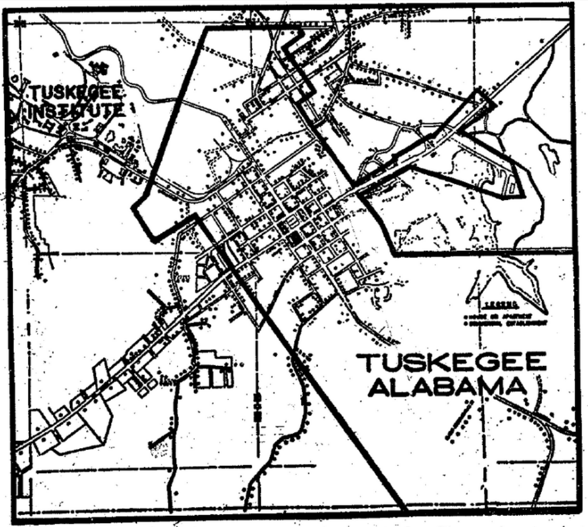
Tradition of Abuse

Draw incumbents in or out

Ohio 1880s

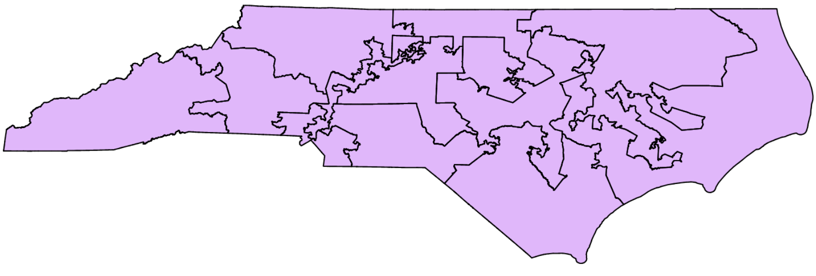
Tuskegee

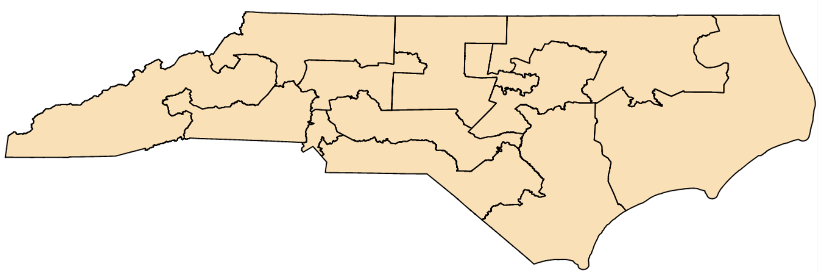
Tuskegee, Alabama



(The entire area of the square comprised the City prior to Act 140. The irregular black-bordered figure within the square represents the post-enactment city.)

Geometry is no longer sufficient





Why?

Better data
Better software
Better ad targeting

But! Public tools have largely caught up with private ones.

The Problem (Modern Version)

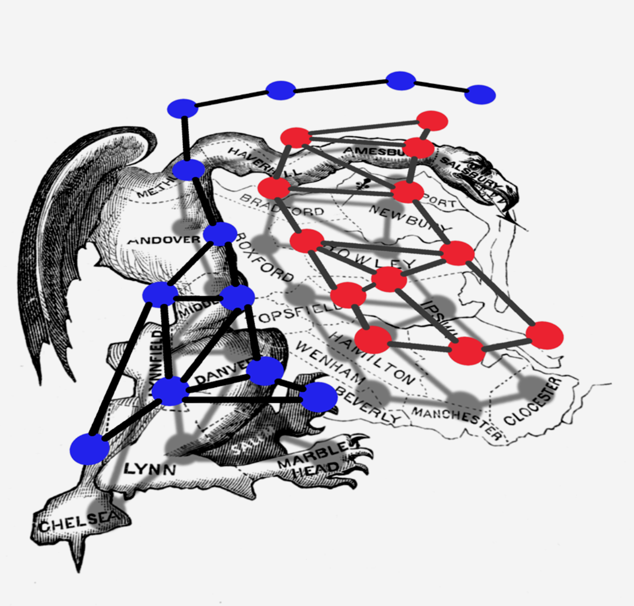
The abundance of data means that its easier to draw a hard-to-detect gerrymander. One way to demonstrate that something is a gerrymander is to show that it is an extreme outlier with respect to some measure we care about. How do we go about demonstrating this?

GRAPHS AND METAGRAPHS

A Graph Theory Problem

Districts are formed from atomic geographic units (Census blocks)
Can we get anything by using the tools of *graph theory*?

Constructing the dual graph



The graph-theoretic definition

A **districting plan** (on a dual graph) is a *partition* of the vertices into *k* disjoint, connected pieces which satisfy the criteria we care about.

equal population

partisan, racial metrics

not splitting towns

Hardness

Unfortunately, no matter how you slice it, redistricting is NP-Hard.

- Population constraints, racial and partisan metrics $\rightarrow$ SUBSETSUM
- Geometric constraints (minimize cut edges, e.g.) $\rightarrow$ MILP
- Optimization problems (find the ‘fairest’ plan ...) $\rightarrow$ *k*-KNAPSACK

However, NP-Hard doesn’t mean impossible.

The Dream

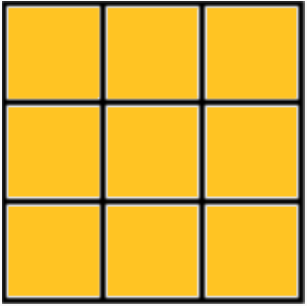
We're working in a very particular corner of the universe of the problem.

Maybe it's easy for our setting?

Does our problem have enough structure that we can just write down all of the plans and look at every one?

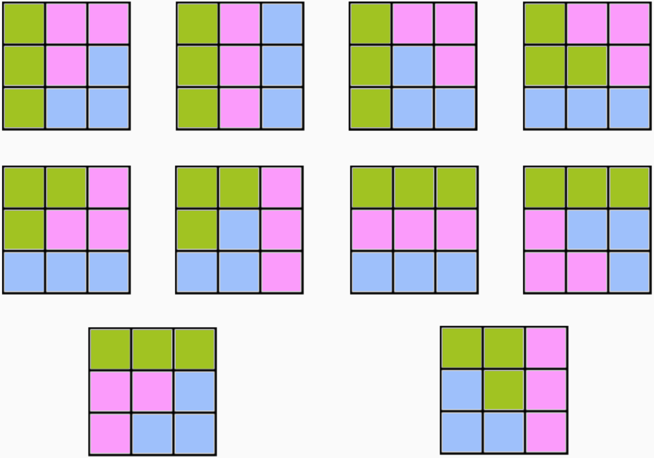
A Combinatorial Warmup

Let's take the fictional state of Gridlandia.



How many ways are there to divide Gridlandia into **3** connected pieces of size **3**?

Gridlandia Enumeration



And larger?

4x4 into 4 pieces of size 4? → 117

5x5 into 5 pieces of size 5? → 4006

6x6 into 6 pieces of size 6? → 451206

7x7 into 7 pieces of size 7? → 158753814 (10^8)

8x8 into 8 pieces of size 8? → 187497290034 (10^{11})

9x9 into 9 pieces of size 9? → 706152947468301 (10^{14})

10x10 into 10 pieces of size 10? → Open Problem!

Since we can't write down all the plans, we need some way of *sampling* them.

Enter The Metagraph

Let's imagine (because we definitely can't write it down) a graph $\mathcal{M}$

There is a vertex for each (valid) districting plan.

The edges are interesting. How do we define 'adjacent'?

Adjacency

We can do whatever we want! Let $f : \mathbb{D} \times [0,1] \to \mathbb{D}$ be any function satisfying the following:

- $\mathrm{Im}(f) = \mathbb{D}$
- f is reversible (i.e. if $f(D_1, \alpha) = D_2$, then there exists β such that $f(D_2, \beta) = D_1$)
- f is efficiently computable

For example, f can be the function ”choose two adjacent districts and a cell in each and try to swap them”

A quick illustration

Lots more stuff at `http://mggg.org/metagraph`

Properties of the metagraph

For our function f, put an edge (D_i, D_j) in the metagraph $\mathcal{M}$ if $f(D_i, \alpha) = D_j$ for some α.

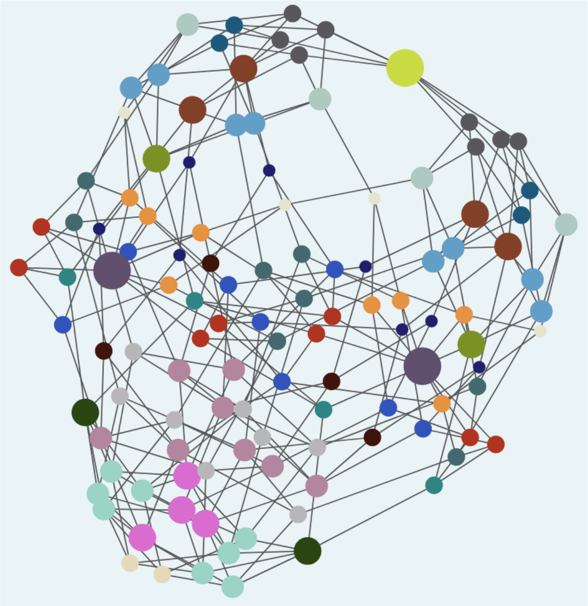
$\mathrm{Im}(f) = \mathbb{D} \rightarrow \mathcal{M}$ is connected

$\rightarrow$ each edge is equipped with a transition probability

f is reversible $\rightarrow \mathcal{M}$ is bidirected

f is efficiently computable $\rightarrow$ we can simulate a random walk

A quick illustration



Random Walks

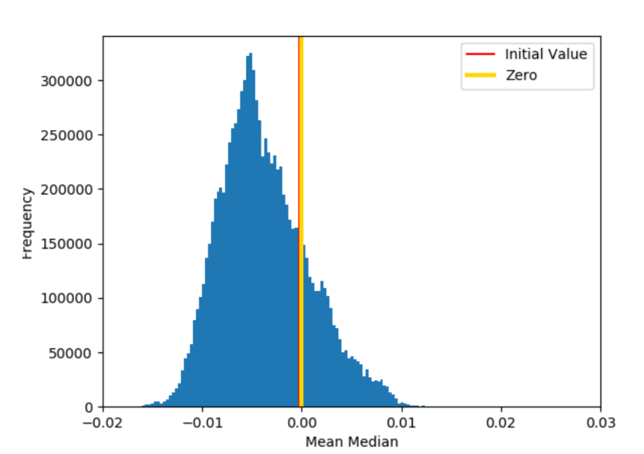
We can define a random walk on the metagraph by starting at some vertex D_1, picking a random α, moving to $f(D_1, \alpha)$, and repeating. This is a **Markov chain**.

So what?

If we run this random walk long enough, the distribution of this sample converges to the **stationary distribution**!

We can pick the stationary distribution by carefully picking the transition probabilities!

How do we use this?



Graphic from DeFord, Duchin & Solomon *Report for the VA House of Delegates*

Proposals

Determining which plans are ”adjacent” affects the sampling procedure. The flip walk

moves very slowly through the space. The flip walk also moves towards plans which are geometrically nonsensical.

Is there a way to move around the space more quickly which avoids both of these?

Recombination

Consider the following proposal function:

pick two adjacent districts in D_1

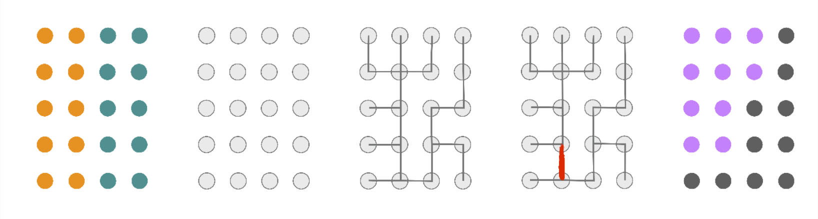
merge them into one ”superdistrict”

find a random spanning tree of the superdistrict

cut the tree in half to make two new districts

let D_2 be the new plan

A quick illustration



Graphic from DeFord, Duchin & Solomon *Report for the VA House of Delegates*

Recombination

Some properties:

- favors generating districts with lots of spanning trees
- population constraints make the choice of cut (roughly) unique
- metagraph nodes have high degree

Experiments with ReCom show that it does, in practice, improve the quality of the sample that the MCMC process generates!

Soon...

There will be nation-wide redistricting at all levels in 2021

Awareness of the problem is higher than ever

How do we get it right?

THANK YOU!